JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **30**, No. 2, May 2017
http://dx.doi.org/10.14403/jcms.2017.30.2.183

# TWO ZAGIER-LIFTS

Soon-Yi Kang*

Abstract. Zagier lift gives a relation between weakly holomorphic modular functions and weakly holomorphic modular forms of weight 3/2. Duke and Jenkins extended Zagier-lifts for weakly holomorphic modular forms of negative-integral weights and recently Bringmann, Guerzhoy and Kane extended them further to certain harmonic weak Maass forms of negative-integral weights. New Zagier-lifts for harmonic weak Maass forms and their relation with Bringmann-Guerzhoy-Kane's lifts were discussed earlier. In this paper, we give explicit relations between the two different lifts via direct computation.

## 1. Introduction

Throughout, $\kappa \in \frac{1}{2}\mathbb{Z}$ and $D$ and $d$ are integers with $D, d \equiv 0, 1 \pmod 4$. We let $\Gamma = SL_2(\mathbb{Z})$ when $\kappa$ is an integer and $\Gamma = \Gamma_0(4)$ when $\kappa \in \mathbb{Z} + \frac{1}{2}$. For a complex number $\tau = x + iy$ with $y > 0$ and $\gamma = \left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right) \in \Gamma$, we define

$$j(\gamma, \tau) := \begin{cases} \sqrt{u\tau + v}, & \text{if } \kappa \in \mathbb{Z}, \\ \left(\frac{u}{v}\right)\varepsilon_v^{-1}\sqrt{u\tau + v}, & \text{if } \kappa \in \frac{1}{2} + \mathbb{Z}, \end{cases}$$

where $\left(\frac{u}{v}\right)$ is the extended Legendre symbol and $\varepsilon_v = 1$ if $v \equiv 1 \pmod 4$ and $\varepsilon_v = i$ if $v \equiv 3 \pmod 4$. Then for any complex-valued function $f$ defined on the upper-half plane $\mathbb{H}$, the weight $\kappa$ slash operator is defined by $f|_\kappa \gamma(\tau) := j(\gamma, \tau)^{-2\kappa} f(\gamma\tau)$. A *weakly holomorphic modular form of weight* $\kappa$ is invariant under the weight $\kappa$ slash operator and holomorphic in $\mathbb{H}$ with possible poles at the cusps. Let $M_\kappa^!$ denote the vector space of weakly holomorphic modular forms of weight $\kappa$ on $\Gamma$. In case of

Received November 28, 2016; Accepted February 28, 2017.
2010 Mathematics Subject Classification: Primary: 11F03, 11F12, 11F30, 11F37.
Key words and phrases: Zagier lift, harmonic weak Maass forms, traces of singular moduli, cycle integral of a modular function.
The author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (NRF-2016R1D1A1B01012258).

$\kappa \in \mathbb{Z} + \frac{1}{2}$, each form in $M^!_\kappa$ satisfies Kohnen's plus space condition, that is, its Fourier expansion is of the form $\sum a(n)q^n$ where $a(n)$ is nonzero only for integers $n$ satisfying $(-1)^{\kappa-1/2}n \equiv 0, 1 \pmod 4$. Here, $q := e(\tau) := e^{2\pi i\tau}$.

Recall that the Fourier expansion of the classical $j$-function is given by

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \cdots \in M^!_0.$$

The $j$-function generates the field of all meromorphic modular functions for $\Gamma$ and its values at quadratic irrationalities in $\mathbb{H}$ are algebraic integers, known as singular moduli. More precisely, for a positive definite quadratic form $Q(x, y) = [a, b, c] = aX^2 + bXY + cY^2$ with discriminant $dD = b^2 - 4ac < 0$ and its associated CM point

$$\tau_Q = \frac{-b + \sqrt{dD}}{2a} \in \mathbb{H},$$

$j(\tau_Q)$ is an algebraic integer in an abelian extension of $\mathbb{Q}(\sqrt{dD})$. When $\mathcal{Q}_d$ denotes the set of positive definite integral binary quadratic forms with discriminant $d$, the group $\Gamma$ acts on $\mathcal{Q}_d$ in the usual way. For each fundamental discriminant $D > 0$ with $dD < 0$ and a $\Gamma$-invariant function $f$ on $\mathbb{H}$, we define the twisted trace of $f$ by

$$\mathrm{Tr}_{d,D}(f) = \sum_{Q \in \Gamma\backslash\mathcal{Q}_{dD}} \chi(Q)\frac{f(\tau_Q)}{|\Gamma_Q|}. \tag{1.1}$$

Here $\Gamma_Q$ is the group of automorphs of $Q$, $\tau_Q$ is the associated CM points, and $\chi$ is an associated genus character which is defined on $\Gamma\backslash Q_{dD}$ by ([5])

$$\chi(Q) = \begin{cases} \chi_D(r), & (a, b, c, D) = 1 \text{ and } (r, D) = 1 \text{ where } Q \text{ represents } r, \\ 0, & (a, b, c, D) > 1, \end{cases}$$

where $\chi_D$ is the Kronecker symbol. We note that if both $d$ and $D$ are fundamental discriminants, $\chi_d(Q) = \chi_D(Q)$.

Let $J = j - 744$ and $\mathrm{Tr}_{d,D}(J)$ be the twisted trace of singular moduli defined in (1.1). In [9], Zagier defined lifts for fundamental discriminants $D > 0$ and $d < 0$ by

(1.2)

$$\begin{aligned} \mathfrak{Z}_d(J) &= q^d && + \textstyle\sum_{D>0} D^{-1/2}\mathrm{Tr}_{d,D}(J)q^D && \in M^!_{\frac{1}{2}}, \\ \mathfrak{Z}_D(J) &= q^{-D} \quad -2\delta_{D,\square} && - \textstyle\sum_{d<0} D^{-1/2}\mathrm{Tr}_{d,D}(J)q^{|d|} && \in M^!_{\frac{3}{2}}, \end{aligned}$$

where $\delta_{D,\square} = 1$ if $D$ is a square and 0 otherwise. The results for $J \in M_0^!$ in (1.2) were generalized to weakly holomorphic modular forms of negative-integral weights by Duke and Jenkins. In [3, Theorem 1], they extended Zagier lifts for $f \in M_{2-2\nu}^!$ ($\nu \geq 2$ an integer) and fundamental discriminant $D$ that

$$\begin{aligned} \mathfrak{Z}_D(f) &\in M_{3/2-\nu}^!, && \text{if } (-1)^\nu D > 0, \\ \mathfrak{Z}_D(f) &\in M_{\nu+1/2}^!, && \text{if } (-1)^\nu D < 0. \end{aligned} \tag{1.3}$$

A *weak Maass form* $h$ of weight $\kappa$ is a smooth function on $\mathbb{H}$ which satisfies:

(i) $h|_\kappa \gamma = h$ for all $\gamma \in \Gamma$,

(ii) $\Delta_\kappa(h) = \lambda h$ where $\Delta_\kappa = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + i\kappa y\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$,

(iii) $h$ has at most linear exponential growth at all cusps of $\Gamma$.

If $\lambda = 0$, then $h$ is called *harmonic* and if $h$ is holomorphic in $\mathbb{H}$ with possible poles at the cusps, then it becomes a weakly holomorphic modular form. Now consider $H_\kappa^!$, the space of weight $\kappa$ harmonic weak Maass forms on $\Gamma$. Again when $\kappa \in \frac{1}{2} + \mathbb{Z}$, each form satisfies the plus space condition.

Bruinier and Funke showed in [2] that the differential operator $\xi_\kappa = 2iy^\kappa \frac{\overline{\partial}}{\partial \overline{\tau}}$ is a surjective map from the space of harmonic weak Maass forms of weight $\kappa$ to the space of weakly holomorphic modular forms of weight $2 - \kappa$. A weight $\kappa$ harmonic weak Maass form has a Fourier expansion at infinity of the form

$$h(\tau) = \sum_{n \gg -\infty} c_h^+(n) q^n + c_h^-(0) y^{1-\kappa} + \sum_{0 \neq n \ll \infty} c_h^-(n) \Gamma(1-\kappa, -4\pi n y) q^n, \tag{1.4}$$

where $\Gamma(a,x) = \int_x^\infty e^{-t} t^{a-1}\, dt$ is the incomplete gamma function. The differential operator $\xi_\kappa$ maps it to

$$\xi_\kappa(h) = (1-\kappa)\overline{c_h^-(0)} - \sum_{0 \neq n \ll \infty} \overline{c_h^-(n)} (-4\pi n)^{1-\kappa} q^{-n} \in M_{2-\kappa}^!. \tag{1.5}$$

We call $\sum_{n<0} c_h^+(n) q^n$ the principal part of the harmonic weak Maass form $h(\tau)$.

The Zagier-lifts in (1.3) have been recently generalized in two different ways. The lifts given by Bringmann, Guerzhoy and Kane in [1] are defined in $H_{2-2\nu}^{cusp}$, the subspace of $H_{2-2\nu}^!$ that consists of harmonic weak Maass forms whose image under the differential operator $\xi_\kappa$ are cusp forms. They contain (1.3) as special cases as they showed that for each $h \in H^{cusp}_{2-2\nu}$,

$$\begin{aligned} \mathfrak{Z}_D(h) &\in H^!_{3/2-\nu}, \quad \text{if } (-1)^\nu D > 0, \\ \mathfrak{Z}_D(h) &\in M^!_{\nu+1/2}, \quad \text{if } (-1)^\nu D < 0. \end{aligned} \tag{1.6}$$

Other lifts were constructed by the author with Jeon and Kim in [7]. They are defined in $H^!_{2-2\nu}$ and extend (1.3) for discriminants not treated in (1.3). For each $h \in H^!_{2-2\nu}$, the new lifts $\mathfrak{Z}^+$ are defined by

$$\begin{aligned} \mathfrak{Z}^+_D(h) &\in H^!_{3/2-\nu}, \quad \text{if } (-1)^\nu D < 0, \\ \mathfrak{Z}^+_D(h) &\in H^!_{\nu+1/2}, \quad \text{if } (-1)^\nu D > 0. \end{aligned} \tag{1.7}$$

In [7], it is simply stated that the two lifts $\mathfrak{Z}_D$ and $\mathfrak{Z}^+_D$ satisfy the following relations: for each $h \in H_{2-2\nu}$, $\xi_{\frac{3}{2}-\nu}(\mathfrak{Z}^+_D(h)) = \mathfrak{Z}_D(h)$ or $\xi_{\frac{3}{2}-\nu}(\mathfrak{Z}_D(h)) = \mathfrak{Z}^+_D(h)$ up to constant ($\nu \neq 2, 3, 4, 5, 7$ in the latter) depending on the sign of corresponding discriminant and parity of $\nu$. In this paper, we give explicit relations between the two lifts via direct computation.

## 2. Whittaker functions

We first recall many properties of Whittaker functions, with which we construct weak Maass-Poincaré Series as in [4, 6]. Whittaker functions $M_{\mu,\nu}(y)$ and $W_{\mu,\nu}(y)$ are linearly independent solutions of the Whittaker differential equation. If $2\nu \notin \mathbb{Z}$, they satisfy that

$$W_{\mu,\nu}(y) = \frac{\Gamma(-2\nu)}{\Gamma(\frac{1}{2}-\nu-\mu)} M_{\mu,\nu}(y) + \frac{\Gamma(2\nu)}{\Gamma(\frac{1}{2}+\nu-\mu)} M_{\mu,-\nu}(y) \tag{2.1}$$

and in particular, $W_{\mu,\nu}(y) = W_{\mu,-\nu}(y)$. Here $\Gamma(z)$ is the gamma function.

The Whittaker functions may have integral representations for certain fixed values of $\mu$ and $\nu$ from which we have that if $\nu - \mu = 1/2$, then

$$M_{\mu,\nu}(y) + (2\mu+1) W_{\mu,\nu}(y) = \Gamma(2\mu+2) y^{-\mu} e^{y/2}, \tag{2.2}$$

and if $\nu + \mu = 1/2$, we have

$$W_{\mu,\nu}(y) = y^{\mu} e^{-y/2}. \tag{2.3}$$

If $\frac{1}{2} - \mu \pm \nu$ is an integer, Whittaker functions can be expressed as an incomplete gamma function $\Gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} dt$. For example, if

$\nu \in \frac{1}{2}\mathbb{Z}$, then

$$W_{\nu-1/2,\nu}(y) = e^{y/2}y^{1/2-\nu}\Gamma(2\nu, y). \tag{2.4}$$

Asymptotic behavior of the Whittaker functions for fixed $\mu$, $\nu$ is also well known:

(2.5)
$$M_{\mu,\nu}(y) \sim \frac{\Gamma(1+2\nu)}{\Gamma(\nu-\mu+\frac{1}{2})}y^{-\mu}e^{\frac{y}{2}} \text{ and } W_{\mu,\nu}(y) \sim y^{\mu}e^{-\frac{y}{2}} \text{ as } y \to \infty,$$

(2.6)
$$M_{\mu,\nu}(y) \sim y^{\nu+\frac{1}{2}} \quad \text{and} \quad W_{\mu,\nu}(y) \sim \frac{\Gamma(2\nu)}{\Gamma(\nu-\mu+\frac{1}{2})}y^{-\nu+\frac{1}{2}} \quad \text{as} \quad y \to 0.$$

Now we define for fixed values $s \in \mathbb{C}$ and $n \in \mathbb{Z}$,

$$\mathcal{M}_{n,\kappa}(y,s) = \begin{cases} \Gamma(2s)^{-1}(4\pi|n|y)^{-\frac{\kappa}{2}}M_{\frac{\kappa}{2}\mathrm{sgn}(n),s-\frac{1}{2}}(4\pi|n|y), & \text{if } n \neq 0, \\ y^{s-\kappa/2}, & \text{if } n = 0. \end{cases}$$

$$\mathcal{W}_{n,\kappa}(y,s)$$
$$= \begin{cases} \Gamma(s+\frac{\kappa}{2}\mathrm{sgn}(n))^{-1}|n|^{\frac{\kappa}{2}-1}(4\pi y)^{-\frac{\kappa}{2}}W_{\frac{\kappa}{2}\mathrm{sgn}(n),s-\frac{1}{2}}(4\pi|n|y), & \text{if } n \neq 0, \\ \frac{(4\pi)^{1-\kappa}y^{1-s-\kappa/2}}{(2s-1)\Gamma(s-\kappa/2)\Gamma(s+\kappa/2)}, & \text{if } n = 0. \end{cases}$$

The function

$$\varphi_{m,\kappa}(z,s) := \mathcal{M}_{m,\kappa}(y,s)e(mx) \tag{2.7}$$

is an eigenfunction of the weight $\kappa$ hyperbolic Laplacian $\Delta_\kappa$ and has eigenvalue $s(1-s)+(\kappa^2-2\kappa)/4$. That is,

$$\Delta_\kappa\varphi_{m,\kappa}(z,s) = \left(s-\frac{\kappa}{2}\right)\left(1-\frac{\kappa}{2}-s\right)\varphi_{m,\kappa}(z,s). \tag{2.8}$$

Also, due to the asymptotic behavior of the Whittaker function given in (2.6),

$$\varphi_{m,\kappa}(z,s) = O(y^{\mathrm{Re}(s)-\kappa/2}) \quad \text{as} \quad y \to 0. \tag{2.9}$$

We are interested in its values at $s = \kappa/2$ and $s = 1-\kappa/2$, for which $\Delta_\kappa\varphi_{m,\kappa}(z,s) = 0$. It follows from the properties of Whittaker functions stated above that

(2.10)
$$\mathcal{W}_{n,\kappa}(y, 1-\kappa/2)$$
$$= e^{-2\pi ny}\begin{cases} n^{\kappa-1}, & \text{if } n > 0, \\ |n|^{\kappa-1}\Gamma(1-\kappa)^{-1}\Gamma(1-\kappa, -4\pi ny), & \text{if } n < 0, \\ \frac{(4\pi)^{1-\kappa}}{\Gamma(2-\kappa)}, & \text{if } n = 0. \end{cases}$$

If we assume $\kappa \leq 1/2$, then we find that

(2.11)

$$\mathcal{M}_{n,\kappa}(y, 1-\kappa/2) = e^{-2\pi ny}\begin{cases} (-1)^{\kappa}\left[\Gamma(1-\kappa)^{-1}\Gamma(1-\kappa, -4\pi ny) - 1\right], & \text{if } n > 0, \\ 1 - \Gamma(1-\kappa)^{-1}\Gamma(1-\kappa, -4\pi ny), & \text{if } n < 0, \\ y^{1-\kappa}, & \text{if } n = 0. \end{cases}$$

Also, we have

$$\mathcal{W}_{n,\kappa}(y, \kappa/2) = e^{-2\pi ny}\begin{cases} \Gamma(\kappa)^{-1}n^{\kappa-1}, & \text{if } n > 0, \\ 0, & \text{if } n \leq 0. \end{cases} \tag{2.12}$$

and

$$\mathcal{M}_{n,\kappa}(y, \kappa/2) = e^{-2\pi ny}\begin{cases} \Gamma(\kappa)^{-1}, & \text{if } n \neq 0, \\ 1, & \text{if } n = 0. \end{cases} \tag{2.13}$$

More detailed proofs of (2.10) through (2.13) and references on Whittaker functions are given in [6, Section 2].

## 3. Weak Maass-Poincaré series

If $\phi : \mathbb{R}^+ \to \mathbb{C}$ is a smooth function satisfying $\phi(y) = O_\varepsilon(y^{1+\varepsilon})$ for any $\varepsilon > 0$ and $\Gamma_\infty$ is the subgroup of translations of $\Gamma$, then the general Poincaré series

$$\begin{aligned} G_m(\tau, \phi) &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\phi|_\kappa \gamma)(\tau) \\ &= \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} e(m\mathrm{Re}(\gamma\tau))\phi(\mathrm{Im}(\gamma\tau)), \quad (m \in \mathbb{Z}) \end{aligned} \tag{3.1}$$

is a smooth $\Gamma$-invariant function on $\mathbb{H}$. Let

$$\phi_{m,s}(y) = \begin{cases} 2\pi|m|^{\frac{1}{2}}y^{\frac{1}{2}}I_{s-\frac{1}{2}}(2\pi|m|y), & m \neq 0, \\ y^s, & m = 0, \end{cases} \tag{3.2}$$

with $I_\alpha$ the usual $I$-Bessel function. Then the Niebur Poincaré series

$$G_m(\tau, s) := G_m(\tau, \phi_{m,s}) \tag{3.3}$$

is defined for $\mathrm{Re}\,(s) > 1$ and satisfies

$$\Delta_0 G_m(\tau, s) = (s - s^2)G_m(\tau, s). \tag{3.4}$$

Remark 3.1. The Niebur Poincaré series denoted by $F_m(s;\tau)$ in [3] and [1] are slightly different with (3.3). Their $\phi_{m,s}(y)$ in (3.2) has $|m|^{s-\frac{1}{2}}$ factor instead of $|m|^{\frac{1}{2}}$. Thus $F_m(s;\tau) = |m|^{s-1}G_m(\tau, s)$.

As each $G_m(\tau, s)$ when $m \neq 0$ has an analytic continuation to $\mathrm{Re}\,(s) > 1/2$, we obtain an infinite family of weight 0 harmonic weak Maass forms $\{G_m(\tau, 1) | m \in \mathbb{Z}\}$.

If $\varphi := \varphi_{m,\kappa}(\tau, s)$ is the function defined in (2.7), then the Poincaré series

$$F_{m,k}(\tau, s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\varphi|_\kappa \gamma)(\tau) \tag{3.5}$$

is $\Gamma$-invariant and it follows from (2.9) that $F_{m,\kappa}(\tau, s)$ converges absolutely and uniformly on compacta for $\mathrm{Re}(s) > 1$. Moreover, $F_{m,\kappa}(\tau, s)$ is an eigenfunction of the Laplacian $\Delta_\kappa$ satisfying

$$\Delta_\kappa F_{m,\kappa}(\tau, s) = \left(s - \frac{\kappa}{2}\right)\left(1 - \frac{\kappa}{2} - s\right) F_{m,\kappa}(\tau, s). \tag{3.6}$$

Note that

$$F_{m,0}(\tau, s) = \Gamma(s)^{-1} G_m(\tau, s) \tag{3.7}$$

and the Laplace operator $\Delta_\kappa$ can be expressed in terms of the differential operator $\xi_\kappa$

$$\Delta_\kappa = -\xi_{2-\kappa} \circ \xi_\kappa. \tag{3.8}$$

The Fourier coefficients of $F_{m,\kappa}(\tau, s)$ can be written in terms of Bessel functions and the generalized Kloosterman sum. If $m, n, c$ are integers with $c$ positive, then the Kloosterman sum is given by
(3.9)
$$K_a(m, n; c) := \begin{cases} \sum_{v(c)^*} e\left(\frac{m\bar{v}+nv}{c}\right), & \text{if } a \in \mathbb{Z}, \\ \sum_{v(c)^*} \left(\frac{c}{v}\right) \varepsilon_v^{2a} e\left(\frac{m\bar{v}+nv}{c}\right) \text{ with } 4|c, & \text{if } a \in \frac{1}{2} + \mathbb{Z}, \end{cases}$$

where the sum runs through the primitive residue classes modulo $c$ and $v\bar{v} \equiv 1 \pmod{c}$. It satisfies the symmetry property

$$K_{\frac{3}{2}}(m, n; c) = -iK_{\frac{1}{2}}(-m, -n; c). \tag{3.10}$$

As $K_{2a+1/2}(m, n; c) = K_{1/2}(m, n; c)$ and $K_{2a+3/2}(m, n; c) = K_{3/2}(m, n; c)$ for any integers $a$, it is often convenient to write

$$K^+(m, n; 4c) = (1 - i)\left(1 + \left(\frac{4}{c}\right)\right) K_{1/2}(m, n; 4c) \tag{3.11}$$

and use it for Kloosterman sums of any half-integral weights. Let $c_{m,\kappa}(n,s)$ be given by

$$c_{m,\kappa}(n,s) = (2\pi i^{-\kappa}) \sum_{c>0} \frac{K_\kappa(m,n,c)}{c}$$

$$\times \begin{cases} |mn|^{\frac{1-\kappa}{2}} J_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right), & mn > 0, \\ |mn|^{\frac{1-\kappa}{2}} I_{2s-1}\left(\frac{4\pi\sqrt{|mn|}}{c}\right), & mn < 0, \\ 2^{\kappa-1}\pi^{s+\frac{\kappa}{2}-1}|m+n|^{s-\frac{\kappa}{2}}(c)^{1-2s}, & mn = 0, m+n \neq 0, \\ 2^{2\kappa-2}\pi^{\kappa-1}\Gamma(2s)(2c)^{1-2s}, & m = n = 0, \end{cases} \tag{3.12}$$

where $J_\alpha$ and $I_\alpha$ are the usual Bessel functions.

Proposition 3.2. [6, Corollary 3.3] *If $\kappa > 2$ and $m$ is an integer, then the Fourier expansion of the Poincaré series $F_{m,k}(\tau, \kappa/2)$ is given by*

$$F_{m,\kappa}(\tau, \kappa/2) = \frac{\delta_m}{\Gamma(\kappa)} q^m + \sum_{0<n\in\mathbb{Z}} \frac{c_{m,\kappa}(n,\kappa/2)}{\Gamma(\kappa)} n^{\kappa-1} q^n$$

$$\in \begin{cases} S_\kappa, & \text{if } m > 0, \\ M_\kappa^!, & \text{if } m < 0, \\ M_\kappa, & \text{if } m = 0, \end{cases}$$

*where $\delta_m = \Gamma(\kappa)$ if $m = 0$ and $\delta_m = 1$ otherwise. The coefficients $c_{m,\kappa}(n,s)$ are defined in (3.12).*

Proposition 3.3. [6, Corollary 3.4] *If $\kappa < 0$ and $m$ is an integer, then the Poincaré series $F_{m,\kappa}(\tau, 1-\kappa/2) \in H_\kappa$ if $m \leq 0$ and $F_{m,\kappa}(\tau, 1-\kappa/2) \in H_\kappa^!$ if $m > 0$. Its Fourier expansion is given by*

$$\begin{aligned} F_{m,\kappa}(\tau, 1-\kappa/2) = {} & \mathfrak{m}_{m,\kappa}(y) q^m + \frac{(4\pi)^{1-\kappa}}{(1-\kappa)\Gamma(1-\kappa)} c_{m,\kappa}(0, 1-\kappa/2) \\ & + \sum_{n>0} c_{m,\kappa}(n, 1-\kappa/2) n^{\kappa-1} q^n \\ & + \sum_{n<0} c_{m,\kappa}(n, 1-\kappa/2) |n|^{\kappa-1} \frac{\Gamma(1-\kappa, -4\pi n y)}{\Gamma(1-\kappa)} q^n, \end{aligned}$$

*where*

$$\mathfrak{m}_{m,\kappa}(y) := \begin{cases} y^{1-\kappa}, & \text{if } m = 0, \\ 1 - \frac{\Gamma(1-\kappa,-4\pi m y)}{\Gamma(1-\kappa)}, & \text{if } m < 0, \\ (-1)^{\kappa-1}\left[1 - \frac{\Gamma(1-\kappa,-4\pi m y)}{\Gamma(1-\kappa)}\right], & \text{if } m > 0, \end{cases}$$

*and the coefficients* $c_{m,\kappa}(n,s)$ *are defined in (3.12).*

When $\kappa = \nu + 1/2$, following [8, p. 250], we employ Kohnen's projection operator $pr_\kappa^+$ to construct a weight $\kappa$ Maass-Poincaré series that satisfies the plus space condition, whose Fourier coefficients are supported on $(-1)^\nu n \equiv 0, 1 \pmod 4$. For each $m$ satisfying $(-1)^\nu m \equiv 0, 1 \pmod 4$ and $Re(s) > 1$, we define the Poincaré series $F_{m,\kappa}^+(\tau, s)$ by

$$F_{m,\kappa}^+(\tau, s) = pr_\kappa^+(F_{m,\kappa}(\tau, s)). \tag{3.13}$$

The function $F_{m,\kappa}^+(\tau, s)$ has weight $\kappa$ for $\Gamma$ and satisfies

$$\Delta_\kappa F_{m,\kappa}^+(\tau, s) = \left(s - \frac{\kappa}{2}\right)\left(1 - \frac{\kappa}{2} - s\right) F_{m,\kappa}^+(\tau, s) \tag{3.14}$$

as $F_{m,\kappa}(\tau, s)$ does.

Proposition 3.4. [6, Theorem 4.4] *Let* $\kappa = \nu + 1/2$ *with* $\nu \in \mathbb{Z}$. *Then for any* $m \in \mathbb{Z}$ *and* $s \in \mathbb{C}$ *satisfying* $(-1)^\nu m \equiv 0, 1 \pmod 4$ *and* $\mathrm{Re}(s) > 1$, *the weight* $\kappa$ *Maass-Poincaré series satisfying the plus space condition* $F_{m,\kappa}^+(\tau, s)$ *has the Fourier expansion*

$$F_{m,\kappa}^+(\tau, s) = \mathcal{M}_{m,\kappa}(y, s)e(mx) + \sum_{(-1)^\nu n \equiv 0,1(4)} b_{m,\kappa}(n, s)\mathcal{W}_{n,\kappa}(y, s)e(nx), \tag{3.15}$$

*where the coefficients* $b_{m,\kappa}(n,s)$ *are given by*

$$\begin{aligned} b_{m,\kappa}(n, s) =& 2\pi i^{-\kappa} \sum_{c>0} \left(1 + \left(\frac{4}{c}\right)\right) \frac{K_\kappa(m, n; 4c)}{4c} \\ & \times \begin{cases} |mn|^{\frac{1-\kappa}{2}} J_{2s-1}\left(\dfrac{\pi\sqrt{|mn|}}{c}\right), & mn > 0, \\ |mn|^{\frac{1-\kappa}{2}} I_{2s-1}\left(\dfrac{\pi\sqrt{|mn|}}{c}\right), & mn < 0, \\ 2^{\kappa-1}\pi^{s+\frac{\kappa}{2}-1}|m+n|^{s-\frac{\kappa}{2}}(4c)^{1-2s}, & mn = 0, m+n \neq 0, \\ 2^{2\kappa-2}\pi^{\kappa-1}\Gamma(2s)(8c)^{1-2s}, & m = n = 0. \end{cases} \end{aligned}$$

Hence it follows from (2.11) and (2.10) that for $m > 0$ and $\kappa < 0$

(3.16)
$$F^+_{m,\kappa}(\tau, 1-\frac{\kappa}{2}) = (-1)^{\kappa-1}q^m + \sum_{\substack{(-1)^\nu n \equiv 0,1(4)\\ n>0}} b_{m,\kappa}(n, 1-\frac{\kappa}{2})n^{\kappa-1}q^n$$
$$+ (-1)^\kappa \frac{\Gamma(1-\kappa, -4\pi m y)}{\Gamma(1-\kappa)} q^m$$
$$+ \sum_{\substack{(-1)^\nu n \equiv 0,1(4)\\ n<0}} b_{m,\kappa}(n, 1-\frac{\kappa}{2})|n|^{\kappa-1}\frac{\Gamma(1-\kappa, -4\pi n y)}{\Gamma(1-\kappa)} q^n.$$

Also, we observe from (3.10) and (3.15) that for any integer $k$

(3.17) $$b_{m,\frac{3}{2}-k}(-n, s) = -b_{-m,\frac{1}{2}+k}(n, s)|mn|^{-\frac{1}{2}+k}.$$

## 4. Definitions of two Zagier lifts

We define Zagier lift $\mathfrak{Z}^+_D$ as follows: Throughout, we let $\nu \in \mathbb{N}_{>1}$. If $D$ is a fundamental discriminant satisfying $(-1)^\nu D > 0$, then

(4.1) $$\mathfrak{Z}^+_D(F_{-m,2-2\nu}(\tau,\nu)) = \begin{cases} F^+_{0,\nu+\frac{1}{2}}(\tau, \frac{\nu}{2}+\frac{1}{4}), & \text{if } m = 0,\\ \sum_{n|m}\left(\frac{D}{n}\right)(m/n)^\nu F^+_{\frac{m^2}{n^2}|D|,\nu+\frac{1}{2}}(\tau, \frac{\nu}{2}+\frac{1}{4}), & \text{if } m \neq 0.\end{cases}$$

If $D$ is a fundamental discriminant satisfying $(-1)^\nu D < 0$, then

(4.2) $$\mathfrak{Z}^+_D(F_{-m,2-2\nu}(\tau,\nu)) = \begin{cases} F^+_{0,3/2-\nu}(\tau, \frac{\nu}{2}+\frac{1}{4}), & \text{if } m = 0,\\ \sum_{n|m}\left(\frac{D}{n}\right)(m/n)^{1-\nu} F^+_{\frac{m^2}{n^2}|D|,\frac{3}{2}-\nu}(\tau, \frac{\nu}{2}+\frac{1}{4}), & \text{if } m \neq 0.\end{cases}$$

In [7], $\mathfrak{Z}^+_D$ (for $(-1)^\nu D > 0$) is defined differently for $\nu = 2, 3, 4, 5, 7$. But in this paper, we deal with the uniformly defined lift as given in (4.1). As Poincaré series $F_{m,2-2\nu}(\tau,\nu)$, $(m \in \mathbb{Z})$ span $H^!_{2-2\nu}$ for integer $\nu > 1$, the Zagier lift $\mathfrak{Z}^+_D$ $((-1)^\nu D > 0)$ gives a function

$$\mathfrak{Z}^+_D : H^!_{2-2\nu} \to M^!_{\nu+\frac{1}{2}}$$

by Proposition 3.2. The lift $\mathfrak{Z}^+_D$ $((-1)^\nu D < 0)$defines a function

$$\mathfrak{Z}^+_D : H^!_{2-2\nu} \to H^!_{\frac{3}{2}-\nu}.$$

In order to define the Zagier lifts in [1], we first define the twisted traces of harmonic weak Maass forms. Recall from (1.1) that for each fundamental discriminant $D > 0$ with $dD < 0$ and a $\Gamma$-invariant function $f$ on $\mathbb{H}$, we define the twisted trace of $f$ by

$$\mathrm{Tr}_{d,D}(f) = \sum_{Q \in \Gamma \backslash \mathcal{Q}_{dD}} \chi(Q) \frac{f(\tau_Q)}{|\Gamma_Q|}. \tag{4.3}$$

The *Maass raising operator* is defined by

$$R_\kappa = 2i \frac{\partial}{\partial \tau} + \frac{\kappa}{y} \tag{4.4}$$

and $R_\kappa$ maps a weight $\kappa$ weak Maass form with eigenvalue $\lambda$ to a weight $\kappa + 2$ weak Maass form with eigenvalue $\lambda + \kappa$ under $\Delta_\kappa$. Hence if we let $R_k^n := R_{k+2(n-1)} \circ \cdots \circ R_{k+2} \circ R_k$, then for $h \in H^!_{2-2\nu}$ with $\nu \geq 1$, $R^{\nu-1}_{2-2\nu}(h)$ is $\Gamma$-invariant.

Now for $(-1)^\nu d > 0$ and $(-1)^\nu D < 0$, define the modified twisted traces of $h \in H^{cusp}_{2-2\nu}$ by

$$\hat{\mathrm{Tr}}_{d,D}(h) := (-1)^{\lfloor \frac{\nu+1}{2} \rfloor} (4\pi)^{1-\nu} |d|^{\frac{\nu-1}{2}} |D|^{\frac{-\nu}{2}} \mathrm{Tr}_{d,D}(R^{\nu-1}_{2-2\nu}(h)) \tag{4.5}$$

and define

$$\hat{\mathrm{Tr}}_{D,d}(h) := -\hat{\mathrm{Tr}}_{d,D}(h).$$

Suppose $h \in H^{cusp}_{2-2\nu}$ has the principal part $\sum_{m<0} c_h^+(m) q^m$. When $(-1)^\nu D < 0$, the $Dth$ Zagier lift of $h$ is defined by

$$\begin{aligned} \mathfrak{Z}_D(h)(\tau) = &\sum_{m>0} c_h^+(-m) \sum_{n|m} \chi_D(n)(m/n)^\nu q^{-\frac{m^2}{n^2}|D|} \\ &+ \sum_{\delta : \delta D < 0} \hat{\mathrm{Tr}}_{\delta,D}(h) q^{|\delta|} \in M^!_{\nu+\frac{1}{2}}. \end{aligned} \tag{4.6}$$

Remark 4.1. In [3, p.576] and [1, eq.(3.4)], they have $m^{2\nu-1}$ instead of $m^\nu$. This difference occurs due to the different definitions of Niebur Poincaré series as mention in Remark 3.1.

On the other hand, for each pair $\delta$, $d$ with $(-1)^\nu \delta > 0$ and $(-1)^\nu d > 0$, the $(\delta, d)th$ twisted trace of a cusp form $f$ with weight $2\nu$ is defined in [1] by

$$\mathrm{Tr}_{\delta,d}(f) := \frac{(-1)^\nu 2^{\nu-2}}{3\sqrt{\pi}} \sum_{Q \in \Gamma \backslash \mathcal{Q}_{d\delta}} (d\delta)^{\frac{1-\nu}{2}} \chi(Q) \int_{\Gamma_Q \backslash S_Q} \frac{f(\tau)}{Q(\tau, 1)^{1-\nu}} d\tau, \tag{4.7}$$

where the geodesic $S_Q$ is defined to be the oriented semi-circle $a|\tau|^2 + b\mathrm{Re}\tau + c = 0$, directed counter-clockwise if $a > 0$ and clockwise if $a < 0$.

For $h \in H_{2-2\nu}^{cusp}$, we may define the modified $(\delta, d)th$ twisted trace of $h$ by

$$\hat{\mathrm{Tr}}_{\delta,d}(h) := (-1)^{\lfloor 1-\frac{\nu}{2}\rfloor}(4\pi)^{1-\nu}|d|^{\frac{\nu-1}{2}}|\delta|^{-\frac{\nu}{2}}\mathrm{Tr}_{\delta,d}(\xi_{2-2\nu}(h)). \tag{4.8}$$

In [1, Theorem 1.1], it is shown that for positive discriminants $d$ and $\delta$,

$$\begin{aligned}\sum_{Q\in\Gamma\backslash\mathcal{Q}_{d\delta}} (d\delta)^{\frac{1-\nu}{2}}\chi(Q)\int_{\Gamma_Q\backslash S_Q}\frac{\xi_{2-2\nu}(h(\tau))}{Q(\tau,1)^{1-\nu}}d\tau \\ = C_\nu \sum_{Q\in\Gamma\backslash\mathcal{Q}_{d\delta}} (d\delta)^{\frac{1}{2}}\chi(Q)\int_{\Gamma_Q\backslash S_Q}\frac{R_{2-2\nu}^{\nu-1}(h(\tau))}{Q(\tau,1)}d\tau,\end{aligned} \tag{4.9}$$

where

$$C_\nu = -\frac{3\Gamma(\frac{\nu+1}{2})}{2^{\nu-1}\Gamma(\nu-\frac{1}{2})\Gamma(\frac{\nu}{2})}.$$

If $h$ has the principal part $\sum_{m<0} c_h^+(m)q^m$ and constant coefficient $c_h^+(0)$, then the $Dth$ Zagier lift of $h$ when $(-1)^\nu D > 0$ is defined by

$$\begin{aligned}\mathfrak{Z}_D(h)(\tau) = &\sum_{m>0} c_h^+(-m)\sum_{n|m}\chi_D(n)n^{\nu-1}q^{-\frac{m^2}{n^2}|D|} \\ &+ \frac{1}{2}L(1-\nu,\chi_D)c_h^+(0) + \sum_{\delta:D\delta<0}\hat{\mathrm{Tr}}_{\delta,D}(h)q^{|\delta|} \\ &+ \sum_{\delta:D\delta>0}\hat{\mathrm{Tr}}_{\delta,D}(h)\Gamma(\nu-\frac{1}{2};4\pi|\delta|y)q^{-|\delta|}.\end{aligned} \tag{4.10}$$

For later use, we define one more notation. For $m \in \mathbb{Z}$ and two discriminants $d, D$ with a non-square $dD$, we define traces for the Niebur Poincaré series $G_{-m}(\tau, s)$ by

(4.11)

$$\widetilde{\mathrm{Tr}}_{d,D}(G_{-m}(\tau,s)) := \begin{cases} \mathrm{Tr}_{d,D}(G_{-m}(\tau,s)), & \text{if } dD<0, \\ 2^{1-s}\Gamma(s/2)^{-2}\Gamma(s)\pi\sqrt{dD}\,\mathrm{Tr}^*_{d,D}(G_{-m}(\tau,s)), & \text{if } d>0 \text{ and } D>0, \end{cases}$$

where for a weight 0 function $f$ and positive discriminants $d$ and $D$,

$$\mathrm{Tr}^*_{d,D}(f) := \frac{1}{2\pi}\sum_{Q\in\Gamma\backslash\mathcal{Q}_{dD}}\chi(Q)\int_{\Gamma_Q\backslash S_Q}\frac{f(\tau)}{Q(\tau,1)}d\tau. \tag{4.12}$$

Suppose that $d$ and $D$ are fundamental discriminants with a non-square $dD$ and $Re(s) > 1$. In case $m \neq 0$, it is known from [3, p.583] and [1, Eq.(6.7)] when $dD < 0$ and [4, proof of Proposition 5] and [1, Eq.(6.7)] when $d, D > 0$ that

$$\begin{aligned}\widetilde{\mathrm{Tr}}_{d,D}(G_{-m}(\tau,s)) = \;& \pi\sqrt{2m}\sqrt[4]{|dD|}\textstyle\sum_{n|m}\frac{\chi_D(n)}{n^{1/2}}\sum_{c\equiv 0\,(4)}\frac{K^+\left(d,\frac{m^2D}{n^2};c\right)}{c} \\ & \times\begin{cases} I_{s-\frac{1}{2}}\left(\frac{4\pi}{c}|\frac{m}{n}|\sqrt{|Dd|}\right), & dD<0, \\ J_{s-\frac{1}{2}}\left(\frac{4\pi}{c}|\frac{m}{n}|\sqrt{Dd}\right), & dD>0.\end{cases}\end{aligned} \tag{4.13}$$

## 5. Main results

In this section, we find explicit relations between the two lifts $\mathfrak{Z}_D$ and $\mathfrak{Z}_D^+$. As $\mathfrak{Z}_D$ is defined only on a harmonic weak Maass form $h \in H_{2-2\nu}^{cusp}$ and $F_{-m,2-2\nu}$ with positive integers $m$ form a basis for $H_{2-2\nu}^{cusp}$, it suffices to compare the values of the lifts on $F_{-m,2-2\nu}(\tau,\nu)$ for positive integers $m$.

It follows from Proposition 3.3 that

$$\begin{aligned}F_{-m,2-2\nu}(\tau,\nu) &= q^{-m} + \frac{(4\pi)^{-1+2\nu}}{(-1+2\nu)!}c_{-m,2-2\nu}(0,\nu) \\ &\quad + \sum_{n>0} c_{-m,2-2\nu}(n,\nu)n^{1-2\nu}q^n - \frac{\Gamma(-1+2\nu,4\pi m y)}{\Gamma(-1+2\nu)}q^{-m} \\ &\quad + \sum_{n<0} c_{-m,2-2\nu}(n,\nu)|n|^{1-2\nu}\frac{\Gamma(-1+2\nu,-4\pi n y)}{\Gamma(-1+2\nu)}q^n.\end{aligned} \tag{5.1}$$

Applying (1.5) to (5.1), we obtain from (3.12) and (3.10) that

$$\begin{aligned}&\xi_{2-2\nu}(F_{-m,2-2\nu}(\tau,\nu)) \\ &= \frac{(4\pi m)^{-1+2\nu}}{(-2+2\nu)!}\left(q^m - \sum_{n>0}\overline{c_{-m,2-2\nu}(-n,\nu)}|mn|^{1-2\nu}q^n\right) \\ &= (4\pi m)^{-1+2\nu}(2\nu-1)F_{m,2\nu}(\tau,\nu) \in S_{2\nu}.\end{aligned} \tag{5.2}$$

We also observe from [1, Eq.(4.9)] that

$$R_\kappa(F_{m,\kappa}(\tau,s)) = (s+\frac{\kappa}{2})F_{m,\kappa+2}(\tau,s),$$

which implies with (3.7) that

(5.3)
$$R_{-2}\circ R_{-4}\circ\cdots\circ R_{2-2\nu}\left(F_{-m,2-2\nu}(\tau,\nu)\right) = (\nu-1)!F_{-m,0}(\tau,\nu) = G_{-m}(\tau,\nu).$$

THEOREM 5.1. *Let $D$ be a fundamental discriminant satisfying $(-1)^\nu D<0$. For positive integers $m$ and odd integer $\nu\geq 2$, it holds*

$$\xi_{\frac{3}{2}-\nu}(\mathfrak{Z}_D^+(F_{-m,2-2\nu}(\tau,\nu))) = -\frac{(-4\pi|D|)^{-\frac{1}{2}+\nu}}{\Gamma(-\frac{1}{2}+\nu)}\mathfrak{Z}_D(F_{-m,2-2\nu}(\tau,\nu)).$$

*Proof.* Note that $D>0$. We first compute $\mathfrak{Z}_D(F_{-m,2-2\nu}(\tau,\nu))$. By (4.6), (4.5) and (5.3), we have that

(5.4)
$$\mathfrak{Z}_D(F_{-m,2-2\nu}(\tau,\nu)) = \sum_{n|m}\chi_D(n)(m/n)^\nu q^{-\frac{m^2}{n^2}|D|} + \sum_{\delta<0}(-1)^{\lfloor\frac{\nu+1}{2}\rfloor}(4\pi)^{1-\nu}|\delta|^{\frac{\nu-1}{2}}|D|^{-\frac{\nu}{2}}\mathrm{Tr}_{\delta,D}(G_{-m}(\tau,\nu))q^{|\delta|}\in M^!_{\nu+\frac{1}{2}}.$$

Hence by (4.13),

(5.5)
$$\mathfrak{Z}_D(F_{-m,2-2\nu}(\tau,\nu)) = \sum_{n|m}\chi_D(n)(\frac{m}{n})^\nu q^{-\frac{m^2}{n^2}|D|} + \sum_{n|m}\chi_D(n)\sum_{\delta:\delta D<0}(-1)^{\lfloor\frac{\nu+1}{2}\rfloor}(4\pi)^{1-\nu}(\sqrt{2}\pi) \times(\frac{m}{n})^{\frac{1}{2}}|\frac{\delta}{D}|^{\frac{\nu}{2}-\frac{1}{4}}\sum_{c\equiv 0\,(4)}\frac{K^+\left(\delta,\frac{m^2D}{n^2};c\right)}{c}I_{\nu-\frac{1}{2}}\left(\frac{\pi m\sqrt{|D\delta|}}{cn}\right)q^{|\delta|}.$$

On the other hand, recall from (4.2) that

$$\mathfrak{Z}_D^+(F_{-m,2-2\nu}(\tau,\nu)) = \sum_{n|m}\left(\frac{D}{n}\right)(m/n)^{1-\nu}F^+_{\frac{m^2}{n^2}|D|,\frac{3}{2}-\nu}\left(\tau,\frac{\nu}{2}+\frac{1}{4}\right)\in H^!_{\frac{3}{2}-\nu}.$$

Applying (1.5) to (3.16) with $m$ replaced by $\frac{m^2}{n^2}|D|$ and $\kappa$ replaced by $\frac{3}{2}-\nu$, we obtain

(5.6)

$$\begin{aligned}
&\xi_{\frac{3}{2}-\nu}(\mathfrak{Z}_D^+(F_{-m,2-2\nu}(\tau,\nu)))\\
&\quad= -\frac{(-4\pi|D|)^{-\frac{1}{2}+\nu}}{\Gamma(\nu-\frac{1}{2})}\sum_{n|m}\left(\frac{D}{n}\right)(\frac{m}{n})^{\nu}q^{-\frac{m^2}{n^2}|D|} - \frac{(4\pi)^{-\frac{1}{2}+\nu}}{\Gamma(\nu-\frac{1}{2})}\sum_{n|m}\left(\frac{D}{n}\right)(\frac{m}{n})^{1-\nu}\\
&\qquad\times\sum_{\substack{(-1)^{1-\nu}\delta\equiv 0,1(4)\\ \delta<0}}\overline{b_{\frac{m^2}{n^2}|D|,\frac{3}{2}-\nu}(\delta,\frac{\nu}{2}+\frac{1}{4})}q^{-\delta}\in M^!_{\frac{1}{2}+\nu}.
\end{aligned}$$

Hence $-\frac{\Gamma(-\frac{1}{2}+\nu)}{(-4\pi|D|)^{-\frac{1}{2}+\nu}}\xi_{\frac{3}{2}-\nu}(\mathfrak{Z}_D^+(F_{-m,2-2\nu}(\tau,\nu)))$ and $\mathfrak{Z}_D(F_{-m,2-2\nu}(\tau,\nu))$ both are $\widetilde{\mathfrak{Z}_D}(F_{-m,2-2\nu}(\tau,\nu))$ defined in [1, Eq.(3.9)]:

$$\widetilde{\mathfrak{Z}_D}(F_{-m,2-2\nu}(\tau,\nu)) = \sum_{n|m}\left(\frac{D}{n}\right)(\frac{m}{n})^{\nu}q^{-\frac{m^2}{n^2}|D|}+O(q).$$

As claimed in [1], there is a unique form of such. So the theorem holds when $\nu$ is odd. Note that when $\nu$ is even, $D<0$ and $\delta>0$, but this case does not occur in (5.6). $\square$

Theorem 5.2. *Let $D$ be a fundamental discriminant satisfying $(-1)^{\nu}D>0$. For positive integers $m$ and even integer $\nu\geq 2$, it holds*
(5.7)

$$\xi_{\frac{3}{2}-\nu}(\mathfrak{Z}_D(F_{-m,2-2\nu}(\tau,\nu))) = -(\nu-\frac{1}{2})\sqrt{4\pi}D^{\nu-\frac{1}{2}}\mathfrak{Z}_D^+(F_{-m,2-2\nu}(\tau,\nu)).$$

*Proof.* For a fundamental discriminant $D$ with $(-1)^{\nu}D>0$ (in fact $D>0$, because $\nu$ is even), we have from (4.1), (3.15), (2.12), (2.13), (4.11) and (4.13) that

(5.8)

$$\begin{aligned}\mathfrak{Z}^{+}_{D}(F_{-m,2-2\nu}(\tau,\nu)) &= \sum_{n|m}\left(\frac{D}{n}\right)(m/n)^{\nu}F^{+}_{\frac{m^2}{n^2}D,\nu+\frac{1}{2}}(\tau,\frac{\nu}{2}+\frac{1}{4})\\ &= \frac{1}{\Gamma(\nu+\frac{1}{2})}\sum_{n|m}\left(\frac{D}{n}\right)(\frac{m}{n})^{\nu}\\ &\quad\times\left\{q^{\frac{m^2}{n^2}D}+\sum_{\substack{(-1)^{\nu}\delta\equiv 0,1(4)\\ \delta>0}} b_{\frac{m^2}{n^2}D,\frac{1}{2}+\nu}(\delta,\frac{\nu}{2}+\frac{1}{4})\delta^{\nu-\frac{1}{2}}q^{\delta}\right\}\\ &= \frac{1}{\Gamma(\nu+\frac{1}{2})}\sum_{n|m}\left(\frac{D}{n}\right)(\frac{m}{n})^{\nu}q^{\frac{m^2}{n^2}D}\\ &\quad+\sum_{\substack{(-1)^{\nu}\delta\equiv 0,1(4)\\ \delta>0}}\frac{(-1)^{\frac{\nu}{2}}D^{-\frac{\nu}{2}}\delta^{\frac{\nu}{2}-\frac{1}{2}}}{\Gamma(\nu+\frac{1}{2})}\widetilde{\mathrm{Tr}}_{\delta,D}(G_{-m}(\tau,\nu))q^{\delta}\in S_{\frac{1}{2}+\nu},\end{aligned}$$

where the last equality holds by [7, Theorem 4.1].

On the other hand, we obtain from (5.1) and (4.10) that

(5.9)

$$\begin{aligned}\mathfrak{Z}_{D}(F_{-m,2-2\nu}(\tau,\nu)) &= \sum_{n|m}\chi_D(n)n^{\nu-1}q^{-\frac{m^2}{n^2}|D|}\\ &\quad+\frac{1}{2}L(1-\nu,\chi_D)\frac{(4\pi)^{-1+2\nu}}{(-1+2\nu)!}c_{-m,2-2\nu}(0,\nu)\\ &\quad+\sum_{\delta:D\delta<0}\hat{\mathrm{Tr}}_{\delta,D}(F_{-m,2-2\nu}(\tau,\nu))q^{|\delta|}\\ &\quad+\sum_{\delta:D\delta>0}\hat{\mathrm{Tr}}_{\delta,D}(F_{-m,2-2\nu}(\tau,\nu))\Gamma(\nu-\frac{1}{2};4\pi|\delta|y)q^{-|\delta|}\in H^{cusp}_{\frac{3}{2}-\nu}.\end{aligned}$$

Hence by (1.5), we have that

$$\begin{aligned}&\xi_{\frac{3}{2}-\nu}(\mathfrak{Z}_{D}(F_{-m,2-2\nu}(\tau,\nu)))\\ &\qquad= -\sum_{\substack{\delta:D\delta>0\\ \delta>0}}\overline{\hat{\mathrm{Tr}}_{\delta,D}(F_{-m,2-2\nu}(\tau,\nu))(4\pi\delta)^{\nu-\frac{1}{2}}}q^{\delta}.\end{aligned}\tag{5.10}$$

Applying (4.8), (4.7), (4.9), (5.3), (4.12), and (4.11) in turn below, we find that

(5.11)

$$\begin{aligned}
&\hat{\mathrm{Tr}}_{\delta,D}(F_{-m,2-2\nu}(\tau,\nu))\\
&\quad= (-1)^{1-\frac{\nu}{2}}(4\pi)^{1-\nu}D^{\frac{\nu-1}{2}}\delta^{-\frac{\nu}{2}}\mathrm{Tr}_{\delta,D}(\xi_{2-2\nu}(F_{-m,2-2\nu}(\tau,\nu)))\\
&\quad= \frac{(-1)^{\frac{\nu}{2}}(4\pi)^{1-\nu}2^{\nu-1}\Gamma(\frac{\nu}{2}+\frac{1}{2})\Gamma(\frac{\nu}{2})}{\sqrt{\pi}\Gamma(\nu-\frac{1}{2})\Gamma(\nu)}D^{\frac{\nu}{2}-\frac{1}{2}}\delta^{-\frac{\nu}{2}}\widetilde{\mathrm{Tr}}_{\delta,D}(G_{-m}(\tau,\nu))\\
&\quad= \frac{(-1)^{\frac{\nu}{2}}(4\pi)^{1-\nu}}{\Gamma(\nu-\frac{1}{2})}D^{\frac{\nu}{2}-\frac{1}{2}}\delta^{-\frac{\nu}{2}}\widetilde{\mathrm{Tr}}_{\delta,D}(G_{-m}(\tau,\nu)).
\end{aligned}$$

It then follows from (5.10) and (5.11) that

(5.12)

$$\begin{aligned}
&\xi_{\frac{3}{2}-\nu}(\mathfrak{Z}_D(F_{-m,2-2\nu}(\tau,\nu)))\\
&\quad= -\sum_{\substack{(-1)^{\nu}\delta\equiv 0,1(4)\\ \delta>0}}\frac{(-1)^{\frac{\nu}{2}}\sqrt{4\pi}}{\Gamma(\nu-\frac{1}{2})}D^{\frac{\nu}{2}-\frac{1}{2}}\delta^{\frac{\nu}{2}-\frac{1}{2}}\widetilde{\mathrm{Tr}}_{\delta,D}(G_{-m}(\tau,\nu))q^{\delta}\in S_{\frac{1}{2}+\nu}.
\end{aligned}$$

Finally, the theorem follows from (5.8) and (5.12). □

## References


[1] K. Bringmann, P. Guerzhoy, and B. Kane, *Shintani lifts and fractional derivatives for harmonic weak Maass forms*, Adv. Math. **255** (2014), 641–671.

[2] J. H. Bruinier and J. Funke, *On two geometric theta lifts*, Duke Math. J. **125** (2004), no. 1, 45–90.

[3] W. Duke and P. Jenkins, *Integral traces of singular values of weak Maass form*, Algebra and Number Theory **2** (2008), no. 5, 573–593.

[4] W. Duke, Ö. Imamoḡlu, and Á. Tóth, *Cycle integrals of the j-function and mock modular forms*, Ann. Math. **173** (2011), no. 2, 947–981.

[5] B. Gross, W. Kohnen, and D. Zagier, *Heegner points and derivatives of L-series, II*, Math. Ann. **278** (1987), 497–562.

[6] D. Jeon, S.-Y. Kang, and C. H. Kim, *Weak Maass-Poincaré series and weight 3/2 mock modular forms*, J. Number Theory **133** (2013), no. 8, 2567–2587.

[7] D. Jeon, S.-Y. Kang, and C. H. Kim, *Zagier-lift type arithmetic in harmonic weak Maass forms*, J. Number Theory **169** (2016), 227–249.

[8] W. Kohnen, *Fourier coefficients of modular forms of half-integral weight*, Math. Ann. **271** (1985), 237–268.

[9] D. Zagier, *Traces of singular moduli*, Motives, polylogarithms and Hodge theory, Part I, Irvine, CA, 1998, International Press Lecture Series 3, Part I (Int. Press, Somerville, MA, 2002) 211-244.

*
Department of Mathematics
Kangwon National University
Chuncheon 24341, Republic of Korea
*E-mail*: sy2kang@kangwon.ac.kr